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A NOTE ON THE RELATIONSHIP BETWEEN STOKES MULTIPLIERS AND FORMAL SOLUTIONS OF ANALYTIC DIFFERENTIAL EQUATIONS*

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Abstract. This paper generalizes a result of Balser, Jurkat, and Lutz [*J. Math. Anal. Appl.*, 71 (1979), pp. 48-94] concerning the relation between the Stokes multipliers of a homogeneous linear differential equation with an irregular singularity and the asymptotic behaviour of the coefficients of the formal solutions of the equation.

Key words. irregular singularity, formal solutions, Stokes multipliers, Mellin transform, Cauchy-Heine transform, difference equation

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0. Introduction. We consider homogeneous linear differential equations of a complex variable x with an irregular singularity at ∞. If (D) is such an equation of order $m\in\mathbb{N}$, it possesses m linearly independent formal solutions of the form

$$\hat{f}_j(t)=\hat{h}_j(t)t^{\rho}\exp q_j(t),\qquad j\in\{1,\cdots,m\},\tag{0.1}$$

where $t=x^{1/p}$ for some $p\in\mathbb{N}$, $\hat{h}_j\in\mathbb{C}[\![t^{-1}]\!][\log t]$, $\rho_j\in\mathbb{C}$, and $q_j\in\mathbb{C}[t]$ for all $j\in\{1,\cdots,m\}$.

In [1], Balser, Jurkat, and Lutz establish a relation between the Stokes multipliers of a particular set of solutions and the asymptotic behaviour of the coefficients of the formal series $\hat{h}_j$ $(j=1,\cdots,m)$, for second-order equations of unit rank. Schäfke (cf. [8]) has derived similar results for a class of first-order differential systems. In this note, these results are generalized to equations of arbitrary order and rank. We use Mellin transforms of solutions of an associated differential equation to represent the coefficients of $\hat{h}_j$ $(j\in\{1,\cdots,m\})$, and Cauchy-Heine transforms to represent solutions of (D) defined below. Our approach is based on the work of Ramis [6] and Duval [4].

1. A preliminary result. Let (D) be a differential equation of the type mentioned in the Introduction, of order m. In this section we introduce a particular system of solutions $\{f_j^{\nu}, j\in\{1,\cdots,m\},\nu\in\mathbb{Z}\}$ of (D), with the property that, for each $j\in\{1,\cdots,m\}$ and each $\nu\in\mathbb{Z}$, f_j^{ν} is represented asymptotically by $\hat{f}_j$ as $t\to\infty$ in a certain sector S_ν. Other systems of solutions could be used instead, such as that discussed in [5, Satz IV′, p. 99]; this discussion would lead to analogous results. The system defined below has a small technical advantage (cf. the remark at the end of § 2).

We begin by introducing some notation. By (D_t) we will denote the equation into which (D) is carried by the change of variable $t=x^{1/p}$. (D_t) possesses m formal solutions of the form (0.1). For all $i,j\in\{1,\cdots,m\}$ we will write

$$\rho_i-\rho_j=\rho_{ij},\quad q_i-q_j=q_{ij},\quad \deg q_{ij}=k(i,j).$$

We will assume that $k(i,j)\neq 0$ if $i\neq j$, in which case $\hat{h}_j\in\mathbb{C}[\![t^{-1}]\!]$ for all $j\in\{1,\cdots,m\}$. Without being essential, this restriction simplifies the argument presented below.

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If $i \neq j$, the leading term of q_{ij} is of the form

$$\lambda_{ij} t^{k(i,j)}, \qquad \lambda_{ij} \in \mathbb{C}^*$$

and we define

$$\theta_{ij}^l = \frac{1}{k(i,j)} \{\arg \lambda_{ij} - (2\lambda + 1)\pi\}, \qquad l \in \mathbb{Z}$$

for some fixed determination of $\arg \lambda_{ij}$. Let J denote the set of all ordered pairs (i,j) with $i, j \in \{1, \cdots, m\}$ and $i \neq j$. To each triplet $(i,j,l) \in J \times \mathbb{Z}$ we will assign an integer $\nu = n(i,j,l)$ and will write

$$\theta_{ij}^l = \theta_\nu \quad \text{and} \quad k(i,j) = k_\nu.$$

We choose $n(i,j,l)$ in such a way that, first, the mapping $n: J \times \mathbb{Z} \to \mathbb{Z}$ is a bijection, and second,

$$\theta_{\nu+1} \leqq \theta_\nu \quad \text{for all } \nu \in \mathbb{Z} \text{ and } k_{\nu+1} \leqq k_\nu \quad \text{whenever } \theta_\nu = \theta_{\nu+1}.$$

Let

$$N = \sum_{(i,j) \in J} k(i,j).$$

Noting that

$$\theta_{ij}^{l+k(i,j)} = \theta_{ij}^l - 2\pi,$$

we readily verify that

$$\theta_{\nu+N} = \theta_\nu - 2\pi. \tag{1.1}$$

We impose the following additional conditions on the mapping n:

$$n(i,j,l+k(i,j)) = n(i,j,l) + N. \tag{1.2}$$

Furthermore, we define

$$\sigma(i,j) = \{n(i,j,l): l \in \mathbb{Z}\}, \qquad (i,j) \in J,$$

$$\sigma(j) = \bigcup_{i \in \{1,\cdots,m\} \setminus \{j\}} \sigma(i,j), \qquad j \in \{1, \cdots, m\},$$

$$J_\nu = \{(i,j) \in J: k(i,j) = k_\nu \text{ and } \theta_{ij}^l = \theta_\nu \text{ for some } l \in \mathbb{Z}\}, \qquad \nu \in \mathbb{Z}.$$

Note that $\nu \in \sigma(i,j)$ implies $(i,j) \in J_\nu$ but that the inverse is not true, as several θ_ν may coincide.

The directions $\arg t = -\theta_\nu - (\pi/2k_\nu)$, $\nu \in \mathbb{Z}$, of the Riemann surface of $\log t$, are the so-called Stokes directions of (D_t). The Stokes directions of (D) are given by $\arg x = -p(\theta_\nu + (\pi/2k_\nu))$, $\nu \in \mathbb{Z}$.

Let α and β be real numbers such that $\alpha < \beta$, and let $\mathbb{C}_\infty$ denote the Riemann surface of $\log t$. By $S(\alpha, \beta)$ we denote the sector

$$S(\alpha, \beta) = \{t \in \mathbb{C}_\infty: \alpha < \arg t < \beta\}.$$

For every $\nu \in \mathbb{Z}$ we define a sector $S_\nu = S(\alpha_\nu, \beta_\nu)$, where

$$\alpha_\nu = -\min_{\nu' < \nu} \left(\theta_{\nu'} + \frac{\pi}{2k_{\nu'}}\right), \qquad \beta_\nu = -\max_{\nu' \geqq \nu} \left(\theta_{\nu'} - \frac{\pi}{2k_{\nu'}}\right). \tag{1.3}$$

Obviously, $\alpha_\nu \leqq \alpha_{\nu+1}$ and $\beta_\nu \leqq \beta_{\nu+1}$. Furthermore, $\alpha_{\nu+N} = \alpha_\nu + 2\pi$ and $\beta_{\nu+N} = \beta_\nu + 2\pi$; hence

$$S_{\nu+N} = e^{2\pi i} S_\nu, \qquad \nu \in \mathbb{Z}.$$

Putting

$$\max_{(i,j)\in J} k(i,j)=\kappa_1,$$

we have

$$\theta_\nu+\frac{\pi}{2\kappa_1}\leqq\min_{\nu'<\nu+1}\left(\theta_{\nu'}+\frac{\pi}{2k_{\nu'}}\right)\leqq\theta_\nu+\frac{\pi}{2k_\nu},$$

$$\theta_\nu-\frac{\pi}{2k_\nu}\leqq\max_{\nu'\geqq\nu}\left(\theta_{\nu'}-\frac{\pi}{2k_{\nu'}}\right)\leqq\theta_\nu-\frac{\pi}{2\kappa_1}.$$

Hence it follows that

$$S\left(-\theta_\nu-\frac{\pi}{2\kappa_1},-\theta_\nu+\frac{\pi}{2\kappa_1}\right)\subset S_\nu\cap S_{\nu+1}\subset S\left(-\theta_\nu-\frac{\pi}{2k_\nu},-\theta_\nu+\frac{\pi}{2k_\nu}\right). \tag{1.4}$$

Any single mth order differential equation is equivalent to a system of first-order differential equations. Let $[D_t]$ denote the system of first-order differential equations corresponding to (D_t). This system possesses a formal fundamental matrix of the form

$$\hat{F}(t)=\hat{H}(t)t^R\exp Q(t), \tag{1.5}$$

where $\hat{H}\in Gl(m;\mathbb{C}[\![t^{-1}]\!])$, $R=\operatorname{diag}\{\rho_1,\cdots,\rho_m\}$, and $Q=\operatorname{diag}\{q_1,\cdots,q_m\}$. We will use the following result, which is an immediate consequence of Theorem 2.1 of [7].

THEOREM 1.6. *Equation* $[D_t]$ *possesses a system of fundamental matrices* $\{\tilde{F}_\nu,\ \nu\in\mathbb{Z}\}$ *with the following properties*:

(i) $\tilde{F}_\nu(t)\sim\hat{F}(t)$ *as* $t\to\infty$ *in* S_ν;

(ii) $\tilde{F}_{\nu+N}(t)=\tilde{F}_\nu(t\,e^{-2\pi i})\exp 2\pi iR$;

(iii) $(\tilde{F}_\nu^{-1}\tilde{F}_{\nu+1}-I)_{ij}\neq 0$ $(i,j\in\{1,\cdots,m\})$ *implies* $(i,j)\in J_\nu$.

Proof. Let $\kappa_1>\kappa_2>\cdots>\kappa_r$ be the different values of $k(i,j)$, $(i,j)\in J$. According to Theorem 2.1 of [7], the matrix $\hat{H}$ in (1.5) can be factorized in the following way:

$$\hat{H}=\hat{H}_1\cdots\hat{H}_r,$$

where $\hat{H}_j\in Gl(m;\mathbb{C}\{t^{-1}\}_{s_j})$, $s_j=1+1/\kappa_j$. (For an explanation of the notation used in this proof we refer the reader to [6].) Let H_j^ν denote the matrix function obtained by analytic continuation on the Riemann surface of $\log t$ of the sum of $\hat{H}_j$ in a direction $\theta\in(\eta_j^\nu,\theta_j^\nu)$, where

$$\eta_j^\nu=\max\{\theta_{\nu'}\colon \nu'\geqq\nu,\ k_{\nu'}=\kappa_j\},\qquad \theta_j^\nu=\min\{\theta_{\nu'}\colon \nu'<\nu,\ k_{\nu'}=\kappa_j,\ \theta_{\nu'}>\eta_j^\nu\},$$

and let

$$\tilde{F}_\nu(t)=H_1^\nu(t)\cdots H_r^\nu(t)t^R\exp Q(t).$$

By Theorem 2.1 of [7], $\tilde{F}_\nu$ is a fundamental system of $[D_t]$. Furthermore, $H_j^\nu(t)\sim\hat{H}_j$ as $t\to\infty$ in

$$S_{j,\nu}\equiv S\left(-\theta_j^\nu-\frac{\pi}{2\kappa_j},-\eta_j^\nu+\frac{\pi}{2\kappa_j}\right).$$

Consequently, $\tilde{F}_\nu(t)\sim\hat{F}(t)$ as $t\to\infty$ in $\bigcap_{j=1}^r S_{j,\nu}$. Obviously, $S_\nu\subset S_{j,\nu}$ for all $j\in\{1,\cdots,r\}$ and thus (i) is satisfied. Due to (1.1), $H_j^{\nu+N}(t)=H_j^\nu(t\,e^{-2\pi i})$ for all $j\in\{1,\cdots,r\}$ and hence it follows that (ii) holds as well. Now let $l\in\{1,\cdots,r\}$ such that $\kappa_l=k_\nu$. For all $j\neq l$ we have

$$\eta_j^{\nu+1}=\eta_j^\nu\leqq\theta_\nu\leqq\theta_j^{\nu+1}=\theta_j^\nu \tag{1.7}$$

and hence $H_j^{\nu+1}=H_j^\nu$. If $J_\nu=J_{\nu+1}$ and $\theta_\nu=\theta_{\nu+1}$, then (1.7) is also true for $j=l$, so we find

$$\tilde{F}_\nu^{-1}\tilde{F}_{\nu+1}=I \quad \text{if } J_\nu=J_{\nu+1} \text{ and } \theta_\nu=\theta_{\nu+1}.$$

Now suppose that $J_\nu\neq J_{\nu+1}$, or $\theta_\nu>\theta_{\nu+1}$. We have

$$\tilde{F}_\nu^{-1}\tilde{F}_{\nu+1}=\exp\{-Q(t)\}t^{-R}(H_r^\nu)^{-1}\cdots(H_l^\nu)^{-1}H_l^{\nu+1}H_{l+1}^\nu\cdots H_r^\nu t^R\exp Q(t).$$

Furthermore,

$$\eta_l^\nu=\theta_\nu=\theta_l^{\nu+1} \tag{1.8}$$

in this case. From (1.7) and (1.8) we deduce that

$$S_{l,\nu}\cap S_{l,\nu+1}=S\left(-\theta_\nu-\frac{\pi}{2k_\nu},-\theta_\nu+\frac{\pi}{2k_\nu}\right)\subset S_{j,\nu}\quad \text{for all } j>l.$$

Now, $H_j^\nu\in Gl(m;A_{s_j}(S_{j,\nu}))$ for all $j\in\{1,\cdots,r\}$ and all $\nu\in\mathbb{Z}$, and thus $(H_l^\nu)^{-1}H_l^{\nu+1}-I\in\operatorname{End}(m;A_{0,s_l}(S_{l,\nu}\cap S_{l,\nu+1}))$. It follows that

$$(H_r^{(\nu)})^{-1}\cdots(H_l^\nu)^{-1}H_l^{\nu+1}\cdots H_r^\nu-I\in\operatorname{End}(m;A_{0,s_l}(S_{l,\nu}\cap S_{l,\nu+1})).$$

Consequently, for all h, $i\in\{1,\cdots,m\}$ we have

$$(\tilde{F}_\nu^{-1}\tilde{F}_{\nu+1}-I)_{hi}\exp q_{hi}\in A_{0,s_l}\left(S\left(-\theta_\nu-\frac{\pi}{2k_\nu},-\theta_\nu+\frac{\pi}{2k_\nu}\right)\right).$$

This implies that either $(\tilde{F}_\nu^{-1}\tilde{F}_{\nu+1}-I)_{hi}=0$, or $\exp q_{hi}\in A_{0,s_l}(S(-\theta_\nu-\pi/2k_\nu,-\theta_\nu+\pi/2k_\nu))$. The second possibility occurs only if $k(h,i)=k_\nu$ and $\operatorname{Re}\lambda_{hi}t^{k(h,i)}<0$ for all $t\in S(-\theta_\nu-\pi/2k_\nu,-\theta_\nu+\pi/2k_\nu)$, i.e., if $(h,i)\in J_\nu$. This completes the proof of Theorem 1.6.

COROLLARY 1.9. *Equation* (D_t) *possesses a system of solutions* $\{f_j^\nu, j\in\{1,\cdots,m\},\nu\in\mathbb{Z}\}$ *with the following properties*:

(i) $f_j^\nu(t)\sim\hat{f}_j(t)$ *as* $t\to\infty$ *in* S_ν;

(ii) $f_j^{\nu+N}(t)=f_j^\nu(t\,e^{-2\pi i})\,e^{2\pi i\rho_j}$;

(iii) *There exist complex numbers* s_ν, $\nu\in\mathbb{Z}$, *such that*

$$f_j^{\nu+1}-f_j^\nu=\begin{cases} s_\nu f_i^\nu & \textit{if } \nu\in\sigma(i,j),\\ 0 & \textit{otherwise.}\end{cases}$$

We will call the numbers s_ν the Stokes multipliers of the system of solutions $\{f_j^\nu, j\in\{1,\cdots,m\},\nu\in\mathbb{Z}\}$. Note that (ii) and (iii) imply that $s_{\nu+N}=s_\nu\,e^{-2\pi i\rho_{ij}}$.

Proof of Corollary 1.9. We will prove the equivalent statement for the system $[D_t]$, i.e., we will prove the existence of fundamental matrices F_ν, $\nu\in\mathbb{Z}$, with the following properties:

(i)′ $F_\nu(t)\sim\hat{F}(t)$ as $t\to\infty$ in S_ν;

(ii)′ $F_{\nu+N}(t)=F_\nu(t\,e^{-2\pi i})\exp 2\pi iR$;

(iii)′ $(F_\nu^{-1}F_{\nu+1}-I)_{ij}\neq0$ implies that $i\neq j$ and $\nu\in\sigma(i,j)$.

Let $\mu\in\mathbb{Z}$ and suppose that $J_{\mu-1}\neq J_\mu=\cdots=J_{\mu+r-1}\neq J_{\mu+r}$ and $\theta_\mu=\theta_{\mu+1}=\cdots\theta_{\mu+r-1}$ for some $r\in\mathbb{N}$. Thus J_μ consists of r pairs (i_h,j_h), $h=1,\cdots,r$, with $k(i_h,j_h)=k_\mu$, and there exist integers l_h such that

$$\theta_{i_hj_h}^{l_h}=\theta_{\mu+h-1}=\theta_\mu,\qquad h\in\{1,\cdots,r\}. \tag{1.10}$$

Consequently, $\alpha_{\mu+1}=\alpha_{\mu+h}$ and $\beta_\mu=\beta_{\mu+h-1}$ for all $h\in\{1,\cdots,r\}$ (cf. (1.3)). If $r>1$ it follows that

$$S_{\mu+h}=S_\mu\cap S_{\mu+r},\qquad h\in\{1,\cdots,r-1\}. \tag{1.11}$$

We take $F_\mu = \tilde{F}_\mu$ and $F_{\mu+r} = \tilde{F}_{\mu+r}$. If $r=1$, (i)′–(iii)′ are automatically satisfied for $\nu=\mu$. Now suppose that $r>1$. Let

$$\tilde{F}_\mu^{-1}\tilde{F}_{\mu+r} = C.$$

According to Theorem 1.6, $(C-I)_{ij}=0$ unless $(i,j)\in J_\mu$, i.e., unless $(i,j)\in \{(i_1,j_1),\cdots,(i_r,j_r)\}$. We readily verify that J_μ is an antisymmetric and transitive set (cf. [5]), i.e., $(i,j)\in J_\mu$ implies $(j,i)\notin J_\mu$, and if J_μ contains both (i,j) and (j,k) then it also contains (i,k). Due to these properties, C can be written uniquely as a product $C=C_1\cdots C_r$, such that $(C_h-I)_{ij}=0$ unless $(i,j)=(i_h,j_h)$, and thus $\mu+h-1\in\sigma(i,j)$ (cf. [5, p. 82]). Hence, if we choose

$$F_{\mu+h}=F_\mu C_1\cdots C_h, \qquad h\in\{1,\cdots,r-1\},$$

then (iii)′ obviously holds for $\nu=\mu+h-1$, $h\in\{1,\cdots,r\}$, and hence for all $\nu\in\mathbb{Z}$. Furthermore, for all $h\in\{1,\cdots,r\}$ and $i,j\in\{1,\cdots,m\}$ we have

$$(F_{\mu+h}-F_{\mu+h-1})_{ij}=\begin{cases}0 & \text{if } j\neq j_h,\\ (F_{\mu+h-1})_{ii_h}(C_h)_{i_hj_h} & \text{if } j=j_h.\end{cases} \tag{1.12}$$

Now suppose that $F_{\mu+h-1}(t)\sim \hat{F}(t)$ as $t\to\infty$ in $S_{\mu+h-1}$. Thus

$$(F_{\mu+h-1})_{ii_h}(t)\sim \hat{H}_{ii_h}(t)t^{\rho_{ih}}\exp q_{i_h}(t), \qquad t\to\infty \text{ in } S_{\mu+h-1}. \tag{1.13}$$

From (1.10) we infer that $\operatorname{Re}\lambda_{i_hj_h}t^{k(i_h,j_h)}<0$ for all $t\in S(-\theta_\mu-(\pi/2k_\mu), -\theta_\mu+(\pi/2k_\mu))$, and hence, in view of (1.4) and (1.11), for all $t\in S_{\mu+h}$, provided $h<r$. This implies that $\exp q_{i_hj_h}(t)$ decreases exponentially as $t\to\infty$ in $S_{\mu+h}$. With (1.13) it follows that

$$(F_{\mu+h-1})_{ii_h}(t)\exp(-q_{j_h}(t))\sim 0 \quad \text{as } t\to\infty \text{ in } S_{\mu+h}.$$

Combining this with (1.12), we conclude that $F_{\mu+h}(t)\sim\hat{F}(t)$ as $t\to\infty$ in $S_{\mu+h}$. By means of induction on h this property can be established for all $h\in\{0,\cdots,r-1\}$ and thus (i)′ is true for all $\nu\in\mathbb{Z}$. As $J_\nu=J_{\nu+N}$ for all $\nu\in\mathbb{Z}$, we have

$$F_{\mu+N}=\tilde{F}_{\mu+N}, \qquad F_{\mu+r+N}=\tilde{F}_{\mu+r+N}$$

if μ is chosen as before. Let

$$\tilde{F}_{\mu+N}^{-1}\tilde{F}_{\mu+r+N}=\tilde{C}.$$

It follows from Theorem 1.6, property (ii), that

$$\tilde{C}=\exp(-2\pi iR)\tilde{F}_\mu^{-1}\tilde{F}_{\mu+r}\exp 2\pi iR. \tag{1.14}$$

Furthermore, $\tilde{C}$ can be written uniquely as a product $\tilde{C}=\tilde{C}_1\cdots\tilde{C}_r$ such that $(\tilde{C}_h-I)_{ij}=0$ unless $\mu+h+N-1\in\sigma(i,j)$, or equivalently (due to (1.2)), $\mu+h-1\in\sigma(i,j)$. With (1.14) it follows that

$$\tilde{C}_h=\exp(-2\pi iR)C_h\exp 2\pi iR, \qquad h\in\{1,\cdots,r\}.$$

Thus, $F_{\mu+h+N}(t)=\tilde{F}_{\mu+N}(t)\exp(-2\pi iR)C_1\cdots C_h\exp 2\pi iR$, and hence, in view of Theorem 1.6, property (ii),

$$F_{\mu+h+N}(t)=\tilde{F}_\mu(te^{-2\pi i})C_1\cdots C_h\exp 2\pi iR=F_{\mu+h}(te^{-2\pi i})\exp 2\pi iR, \qquad h\in\{1,\cdots,r\}.$$

This proves (ii)′ for all $\nu\in\mathbb{Z}$.

2. An integral representation for the coefficients of $\hat{h}_j$. Let $j\in\{1,\cdots,m\}$. If $\hat{f}_j$ is a formal solution of (D_t) of the form (0.1), then (under the assumption made in § 1)

$\hat{h}_j$ is a formal power series solution of the equation (D_t^j) resulting from a change of the unknown y in (D_t) to z, where

$$z(t)=y(t)t^{-\rho_j}\exp(-q_j(t)).$$

Throughout this section it will be assumed that (D) has no singularities in the finite complex plane but at most a regular one at the origin. In that case, both (D_t) and (D_t^j) have at most a regular singularity at the origin of the complex t-plane. (D_t^j) may be written in the following form:

$$D_t^j z \equiv \sum_{l=0}^{m}\sum_{h=0}^{N_j} a_{hl}t^h\left(t\frac{d}{dt}\right)^l z=0,$$

where $N_j=\sum_{i\in\{1,\cdots,m\}:i\neq j} k(i,j)$, $a_{hl}\in\mathbb{C}$, $a_{0m}\neq 0$, and $a_{N_j1}\neq 0$. Let

$$\hat{h}_j=\sum_{n=0}^{\infty}\hat{h}_{jn}t^{-n},\qquad \hat{h}_{j0}=1.$$

The coefficients $\hat{h}_{jn}$ with $n\geqq 1$ can be determined by means of a recursive relation, or equivalently, by solving an N_jth-order difference equation (Δ^j) of the form

$$(\Delta^j)\qquad \sum_{h=0}^{N_j}\sum_{l=0}^{m} a_{hl}(-1)^l(n+h)^l y_{n+h}=0,$$

subject to N_j initial conditions.

For each $\nu\in\mathbb{Z}$ let ψ_ν denote the function defined by

$$\psi_\nu(t)=f_i^\nu(t)t^{-\rho_j}\exp(-q_j(t))\quad \text{if } \nu\in\sigma(i,j), \tag{2.1}$$

where f_i^ν is one of the solutions of (D_t) mentioned in Corollary 1.9. Obviously, ψ_ν is a solution of (D_t^j), represented asymptotically by

$$\hat{h}_i(t)t^{\rho_{ij}}\exp q_{ij}(t)$$

as $t\to\infty$ in S_ν. Let γ_ν be a half-line in $S_\nu\cap S_{\nu+1}$ starting from zero. Due to (1.4),

$$\operatorname{Re}\lambda_{ij}t^{k(i,j)}<0\quad \text{for all } t\in S_\nu\cap S_{\nu+1},\qquad \nu\in\sigma(i,j).$$

Therefore, ψ_ν decreases exponentially as $t\to\infty$ on γ_ν. Furthermore, since the origin is at most a regular singular point of (D_t^j), $\psi_\nu(1/t)$ has at most polynomial growth as $t\to 0$. Hence the integral

$$\hat{h}^\nu(n)\equiv -\frac{1}{2\pi i}\int_{\gamma_\nu}\psi_\nu(t)t^{n-1}\,dt \tag{2.2}$$

exists for sufficiently large n. With the use of partial integration it is easily verified that the function $\hat{h}^\nu$ defined by (2.2) satisfies the difference equation (Δ^j) if $\nu\in\sigma(j)$ and n is larger than some integer n_0. Let

$$\tilde{\sigma}(j)=\sigma(j)\cap\{1,\cdots,N\}.$$

For each $j\in\{1,\cdots,m\}$ the N_j functions $\hat{h}^\nu$ with $\nu\in\tilde{\sigma}(j)$ are linearly independent (cf. the remark below) and, consequently, form a fundamental system of solutions of (Δ^j). Hence there exist complex numbers c_ν, $\nu\in\{1,\cdots,N\}$ such that, for each $j\in\{1,\cdots,m\}$,

$$\hat{h}_{jn}=\sum_{\nu\in\tilde{\sigma}(j)}c_\nu\hat{h}^\nu(n),\qquad n\geqq n_0. \tag{2.3}$$

Remark. Let $i, j \in J$ and $l \in \mathbb{Z}$ such that $\nu \equiv n(i,j,l) \in \{1, \cdots, N\}$. With the aid of the saddle-point method it can be shown that $\hat{h}^\nu$ admits an asymptotic representation of the form

$$(2.4) \qquad \hat{h}^\nu(n) \sim C_\nu \Gamma\left(\frac{n+\rho_{ij}}{k_\nu}\right) \exp\left(p_\nu(n+\rho_{ij})\right)(1+g_\nu(n)), \qquad n \to \infty,$$

where p_ν is a polynomial in n^{1/k_ν} of degree not exceeding k_ν, without constant term, $C_\nu \in \mathbb{C}^*$, and $g_\nu \in n^{-1/k_\nu}\mathbb{C}[\![n^{-1/k_\nu}]\!]$. p_ν and C_ν are completely determined by q_{ij} and l and, conversely, p_ν determines q_{ij} and l (cf. [2], [3]). (Due to (1.4), the sector $S_\nu \cap S_{\nu+1}$ contains a saddle point of the function $q_{ij}(t)+n \log t$ if n is sufficiently large, and (2.4) can be proved by a straightforward application of the saddle-point method. For other systems of solutions, such as that discussed in [5], the proof of (2.4) is slightly more involved, as it requires a study of the asymptotic behaviour of ψ_ν outside S_ν.)

The assumption in § 1 that $k(h,i) \neq 0$ for all $(h,i) \in J$ implies that $q_{hj} \neq q_{ij}$ if $h \neq i$. Therefore, if $j \in \{1, \cdots, m\}$ is given, q_{ij} and l determine ν. Now suppose that ν_1, $\nu_2 \in \tilde{\sigma}(j)$ and $\nu_1 \neq \nu_2$. Then it follows that $p_{\nu_1} \neq p_{\nu_2}$ and (2.4) shows that $\hat{h}^{\nu_1}$ and $\hat{h}^{\nu_2}$ must be linearly independent.

3. A relation between the Stokes multipliers s_ν and the coefficients of $\hat{h}_j$. We begin by assuming, as we did in the previous section, that (D) has no singularities in the finite complex plane but at most a regular one at the origin. Under this condition we have the following generalization of the result of Balser, Jurkat, and Lutz [1] mentioned in the Introduction.

THEOREM 3.1. *For all $\nu \in \{1, \cdots, N\}$ the coefficients c_ν in* (2.3) *are equal to the Stokes multipliers s_ν defined in Corollary* 1.9.

Proof. Consider the function h^ν defined by

$$h^\nu(t) = t^{-n_0+1} \int_{\gamma_\nu} \frac{\psi_\nu(\tau)\tau^{n_0-1}}{2\pi i(\tau - t)}\, d\tau, \qquad \nu \in \{1, \cdots, N\},$$

where n_0, γ_ν, and ψ_ν have been defined in the previous section. (The function $h^\nu(t)t^{n_0-1}$ is a Cauchy-Heine transform of $\psi_\nu(t)t^{n_0-1}$.) h^ν is analytic in $\mathbb{C} \backslash \operatorname{Im} \gamma_\nu$. As γ_ν is an arbitrary half-line in $S_\nu \cap S_{\nu+1} = S(\alpha_{\nu+1}, \beta_\nu)$ (cf. (1.3)), starting from zero, h^ν can be continued analytically to the sector

$$\mathbf{S}_\nu \equiv S(\alpha_{\nu+1}, \beta_\nu + 2\pi)$$

by continuously changing the direction of γ_ν. It follows from (2.2) and Proposition 4.2 of [6] that h^ν admits the asymptotic representation

$$(3.2) \qquad h^\nu(t) \sim \sum_{n=n_0}^{\infty} \hat{h}^\nu(n) t^{-n}, \qquad t \to \infty \text{ in } \mathbf{S}_\nu$$

(and even that h^ν is Gevrey of order $1+1/k_\nu$). Furthermore, it is easily seen that

$$(3.3) \qquad h^\nu(t) - h^\nu(t\, e^{2\pi i}) = \psi_\nu(t), \qquad t \in S_\nu \cap S_{\nu+1}.$$

For each $j \in \{1, \cdots, m\}$ let h_j^ν be defined by

$$h_j^\nu(t) = \sum_{n=0}^{n_0-1} \hat{h}_{jn} t^{-n} + \sum_{\mu \in \tilde{\sigma}(j):\mu<\nu} s_\mu h^\mu(t) + \sum_{\mu \in \tilde{\sigma}(j):\mu \geqq \nu} s_\mu h^\mu(t\, e^{2\pi i}),$$

where s_μ, $\mu = 1, \cdots, N$, are the Stokes multipliers mentioned in Corollary 1.9. From (3.2) and (3.3) we conclude that h_j^ν is analytic on $\mathbb{C}_\infty$ and admits the following asymptotic representation:

$$h_j^\nu(t) \sim \sum_{n=0}^{n_0-1} \hat{h}_{jn} t^{-n} + \sum_{n=n_0}^{\infty} \sum_{\mu\in\tilde{\sigma}(j)} s_\mu \hat{h}^\mu(n) t^{-n} \tag{3.4}$$

as $t\to\infty$ in

$$\bigcap_{\mu\in\tilde{\sigma}(j):\mu<\nu} \mathbf{S}_\mu \bigcap_{\mu\in\tilde{\sigma}(j):\mu\geqq\nu} e^{-2\pi i}\mathbf{S}_\mu = S_\nu.$$

Moreover, we have

$$h_j^{\nu+1}(t) - h_j^\nu(t) = \begin{cases} 0 & \text{if } \nu\notin\tilde{\sigma}(j), \quad \nu<N, \\ s_\nu(h^\nu(t) - h^\nu(t\,e^{2\pi i})) = s_\nu\psi_\nu(t) & \text{if } \nu\in\tilde{\sigma}(j), \quad \nu<N, \end{cases}$$

$$h_j^1(t\,e^{-2\pi i}) - h_j^N(t) = \begin{cases} 0 & \text{if } N\notin\sigma(j), \\ s_N(h^N(t) - h^N(t\,e^{2\pi i})) = s_N\psi_N(t) & \text{if } N\in\sigma(j). \end{cases}$$

Putting

$$h_j^\nu(t) t^{\rho_j} \exp q_j(t) = g_j^\nu(t),$$

and using (2.1) and Corollary 1.9, we find the following relations:

$$g_j^{\nu+1} - g_j^\nu = f_j^{\nu+1} - f_j^\nu, \qquad \nu = 1, \cdots, N-1,$$
$$g_j^1(t\,e^{-2\pi i})\, e^{2\pi i\rho_j} - g_j^N(t) = f_j^1(t\,e^{-2\pi i})\, e^{2\pi i\rho_j} - f_j^N(t),$$

or equivalently,

$$\begin{aligned} &g_j^{\nu+1} - f_j^{\nu+1} = g_j^\nu - f_j^\nu, \qquad \nu = 1, \cdots, N-1, \\ &\{g_j^1(t\,e^{-2\pi i}) - f_j^1(t\,e^{-2\pi i})\}\, e^{2\pi i\rho_j} = g_j^N(t) - f_j^N(t). \end{aligned} \tag{3.5}$$

These identities show that $(g_j^1(t) - f_j^1(t))t^{-\rho_j}$ is analytic in $\mathbb{C}\backslash\{0\}$. From the asymptotic properties of g_j^ν and f_j^ν given by (3.4) and Corollary 1.9, in combination with (3.5), we deduce that

$$(g_j^1(t) - f_j^1(t))t^{-\rho_j} \exp(-q_j(t)) \sim \sum_{n=n_0}^{\infty} a_j(n) t^{-n}, \qquad t\to\infty \text{ in } \mathbb{C},$$

where $a_j(n) = \sum_{\nu\in\tilde{\sigma}(j)} s_\nu \hat{h}^\nu(n) - \hat{h}_{jn} = \sum_{\nu\in\tilde{\sigma}(j)} (s_\nu - c_\nu)\hat{h}^\nu(n)$ (cf. (2.3)). Consequently, $\sum_{n=n_0}^\infty a_j(n)t^{-n}$ is a convergent power series and its sum is $(g_j^1(t) - f_j^1(t))t^{-\rho_j} \exp(-q_j(t))$. In view of (2.4) this implies that $s_\nu - c_\nu = 0$ for all $\nu\in\tilde{\sigma}(j)$ (and hence $g_j^\nu = f_j^\nu$ for all $\nu\in\{1, \cdots, N\}$).

If (D) has an irregular singularity at the origin, or other finite singular points, the argument presented above is no longer valid. For example, the integral on the right-hand side of (2.2) may not exist. We therefore modify the path γ_ν in the following way. Let $t_\nu\in S_\nu\cap S_{\nu+1}$, such that all finite singularities of (D_t) are contained within the disk $\{t\in\mathbb{C}: |t|<|t_\nu|\}$ and let $\tilde{\gamma}_\nu$ be a half-line in $S_\nu\cap S_{\nu+1}$, starting from t_ν. Now the integral

$$\hat{h}^\nu(n) \equiv -\frac{1}{2\pi i}\int_{\tilde{\gamma}_\nu} \psi_\nu(t) t^{n-1}\, dt \tag{3.6}$$

is well defined for all $n>0$ and admits the asymptotic representation (2.4) (which is independent of t_ν). Furthermore, let

$$h^\nu(t) = t\int_{\tilde{\gamma}_\nu} \frac{\psi_\nu(\tau)\tau^{-1}}{2\pi i(\tau - t)}\, d\tau.$$

h^ν is analytic in the "sector" $\tilde{\mathbf{S}}_\nu=\{t\in\mathbf{S}_\nu: |t|>|t_\nu|\}$. According to Proposition 4.2 of [6], h^ν is Gevrey of order $1+1/k_\nu$ and is represented asymptotically by

$$\sum_{n=0}^{\infty} \hat{h}^\nu(n)t^{-n}$$

as $t\to\infty$ in $\mathbf{S}_\nu$. Moreover, we have

$$h^\nu(t)-h^\nu(t\,e^{2\pi i})=\psi_\nu(t),\quad t\in S_\nu\cap S_{\nu+1},\quad |t|>|t_\nu|.$$

Defining, for each $j\in\{1,\cdots,m\}$, a function h_j^ν by

$$h_j^\nu(t)=\sum_{\mu\in\tilde{\sigma}(j):\mu<\nu} s_\mu h^\mu(t)+\sum_{\mu\in\tilde{\sigma}(j):\mu\geqq\nu} s_\mu h^\mu(t\,e^{2\pi i})$$

and proceeding as in the proof of Theorem 3.1, we conclude that the function a_j^ν defined by

$$a_j^\nu(t)\equiv f_j^\nu(t)t^{-\rho_j}\exp(-q_j(t))-h_j^\nu(t),\quad j\in\{1,\cdots,m\},\quad \nu\in\{1,\cdots,N\},$$

is holomorphic at ∞ and independent of ν. Thus we finally obtain the following result.

THEOREM 3.7. *For all $j\in\{1,\cdots,m\}$ the coefficients of the formal power series $\hat{h}_j$ can be written in the form*

$$\hat{h}_{jn}=\sum_{\nu\in\tilde{\sigma}(j)} s_\nu\hat{h}^\nu(n)+a_{jn},\qquad n\in\mathbb{N},\tag{3.8}$$

where the numbers s_ν are the Stokes multipliers defined in Corollary 1.9, *the a_{jn} are complex numbers with the property that* $\limsup_{n\to\infty}|a_{jn}|^{1/n}<\infty$, *and $\hat{h}_\nu$ is given by* (3.6) *and represented asymptotically by* (2.4) *as $n\to\infty$.*

In general, for a given $j\in\{1,\cdots,m\}$, one of the functions $\hat{h}_\nu$ with $\nu\in\tilde{\sigma}(j)$ will dominate the others as $n\to\infty$. With the aid of (3.8) and (2.4), the corresponding Stokes multipliers s_ν can be determined from the asymptotic behaviour of $\hat{h}_{jn}$. In many cases we may even do a little better, due to the fact that the Stokes multipliers are analytic functions of certain coefficients of the equation. In particular, if $k(i,j)$ is constant for all $(i,j)\in J$ ("one-leveled" equation), it is possible to determine all Stokes multipliers, if the asymptotic behaviour of $\hat{h}_{jn}$ as a function of these coefficients is known. This is illustrated by the example in § 4.

4. An example. We illustrate the foregoing with a very simple example:

$$(D)\qquad x^2y''(x)+\sigma xy'(x)=(x^4+\alpha x^3+\beta x^2+\gamma x+\delta)y(x),\qquad \sigma,\alpha,\beta,\gamma,\delta\in\mathbb{C}.$$

This equation has two linearly independent formal solutions $\hat{f}_1$ and $\hat{f}_2$ of the form

$$\hat{f}_1(x)=\hat{h}_1(x)x^\rho\exp q(x),\qquad \hat{f}_2(x)=\hat{h}_2(x)x^{-\rho-\sigma-1}\exp(-q(x)),$$

where $q(x)=\frac{1}{2}(x^2+\alpha x)$, $\rho=\frac{1}{2}(\beta-\frac{1}{4}\alpha^2-\sigma-1)$, and $\hat{h}_j(x)=1+\sum_{n=1}^\infty \hat{h}_{jn}x^{-n}$, $j=1,2$.

Using the notation introduced in the previous sections we have $p=1$, $k(1,2)=k(2,1)=2$, $\lambda_{12}=1$, and $\lambda_{21}=-1$. Hence, choosing $\theta_0=0$, we find

$$\theta_\nu=-\frac{\nu\pi}{2},\quad S_\nu=S\left(\frac{\nu\pi}{2}-\frac{3\pi}{4},\frac{\nu\pi}{2}+\frac{\pi}{4}\right),\quad \nu\in\mathbb{Z}.$$

In this particular ("one-leveled") case, (D) possesses two unique solutions, f_1^1 and f_1^3, represented asymptotically by $\hat{f}_1$ as $x\to\infty$ in $S_1\cup S_2$ and $S_3\cup S_4$, respectively, and two unique solutions f_2^2 and f_2^4 represented asymptotically by $\hat{f}_2$ as $x\to\infty$ in $S_2\cup S_3$ and $S_4\cup S_5$, respectively.

The substitution

$$y(x)=z(x)x^{\rho}\exp q(x)$$

takes (D) into

$$(D^1)\qquad x^2z''(x)+\{2x^3+\alpha x^2+(2\rho+\sigma)x\}z'(x)+\{(\alpha\rho+\tfrac{1}{2}\alpha\sigma-\gamma)x+\rho(\rho+\sigma-1)-\delta\}z(x)=0.$$

The coefficients $\hat{h}_{1n}$ satisfy the difference equation

$$(\Delta^1)\qquad 2(n+2)y_{n+2}+\left\{\alpha\left(n-\rho-\frac{\sigma}{2}+1\right)+\gamma\right\}y_{n+1}-\{(n-\rho)(n-\rho-\sigma+1)-\delta\}y_n=0$$

with initial conditions

$$y_0=1,\qquad y_1=\tfrac{1}{2}\left(\alpha\rho+\alpha\frac{\sigma}{2}-\gamma\right). \tag{4.1}$$

For sufficiently large n, according to (2.2), (2.3) and Theorem 3.1, h_{1n} may be written in the form

$$\hat{h}_{1n}=-\frac{s_2}{2\pi i}\int_0^{\infty e^{\pi i}}\psi_2(x)x^{n-1}\,dx-\frac{s_4}{2\pi i}\int_0^{\infty e^{2\pi i}}\psi_4(x)x^{n-1}\,dx,$$

where $\psi_\nu(x)\equiv f_2^\nu(x)x^{-\rho}\exp(-q(x))$, $\nu=2,\ 4$. The asymptotic behaviour of $\hat{h}_{1n}$ is given by

$$\begin{aligned}\hat{h}_{1n}=\frac{1}{4\pi i}\exp\left\{\frac{\alpha^2}{8}-\pi i(2\rho+\sigma)\right\}\Gamma\left(\frac{n-2\rho-\sigma-1}{2}\right)\\ \cdot\left[s_2(-1)^n\exp\left(\alpha\sqrt{\frac{n}{2}}\right)(1+o(1))\right.\\ \left.-s_4\exp\left\{-\pi i(2\rho+\sigma)-\alpha\sqrt{\frac{n}{2}}\right\}(1+o(1))\right],\quad n\to\infty.\end{aligned} \tag{4.2}$$

Suppose, for example, that $\operatorname{Re}\alpha>0$. Then it follows that

$$s_2=4\pi i\exp\left\{\pi i(2\rho+\sigma)-\frac{\alpha^2}{8}\right\}\lim_{n\to\infty}\hat{h}_{1n}\Gamma\left(\frac{n-2\rho-\sigma-1}{2}\right)^{-1}(-1)^n\exp\left(-\alpha\sqrt{\frac{n}{2}}\right). \tag{4.3}$$

If, on the other hand, $\operatorname{Re}\alpha<0$, we have

$$s_4=-4\pi i\exp\left\{2\pi i(2\rho+\sigma)-\frac{\alpha^2}{8}\right\}\lim_{n\to\infty}\hat{h}_{1n}\Gamma\left(\frac{n-2\rho-\sigma-1}{2}\right)^{-1}\exp\left(\alpha\sqrt{\frac{n}{2}}\right). \tag{4.4}$$

As both Stokes multipliers are entire functions of α, once their values for $\operatorname{Re}\alpha>0$ or $\operatorname{Re}\alpha<0$ are known, other values may be found by analytic continuation. Unfortunately, it is not easy to determine the Stokes multipliers as functions of the coefficients of the equation. In general, (4.3) and (4.4) will merely yield approximations of s_2 and s_4 for given values of these coefficients. (In this connection it might be worthwhile to study the parameter dependence of the solutions of the difference equation (Δ^1).) To conclude this section we mention two particular cases in which (Δ^1) can be solved explicitly.

(i) $\alpha=\gamma=0$. Then we have

$$\hat{h}_{1\,2n}=\frac{\Gamma(n-a)\Gamma(n-b)}{n!\,\Gamma(-a)\Gamma(-b)},\quad \hat{h}_{1\,2n+1}=0,\quad n\in\mathbb{N},$$

where a and b are the roots of the equation $(2x-\rho)(2x-\rho-\sigma+1)-\delta=0$. With (4.2) we find

$$s_2=s_4\exp(-\pi i\beta)=-\frac{2\pi i\exp\pi i\beta}{\Gamma(-a)\Gamma(-b)}.$$

(ii) $\sigma=\gamma=\delta=0$. We readily verify that in this case $\hat{h}_{1n}$ may be represented by

$$\hat{h}_{1n}=\frac{1}{\sqrt{\pi}}\binom{n-\rho-1}{n}\int_{-\infty}^{\infty}e^{-(x+\alpha/2)^2}x^n\,dx. \tag{4.5}$$

Comparison of the asymptotic behaviour of the right-hand side of (4.5) with (4.2) yields the following expressions for the Stokes multipliers s_2 and s_4:

$$s_2=-s_4\exp(-2\pi i\rho)=\frac{i\sqrt{\pi}\,2^{-\rho}\exp(2\pi i\rho-\frac{1}{4}\alpha^2)}{\Gamma(-\rho)},$$

in agreement with the result found by Sibuya (cf. [9, Thm. 22.2]).

Remark. After this paper was completed I received a preprint by M. Loday in which she presents a method for computing the Birkhoff invariants of differential systems of order 2 (cf. [10]). Following her approach, we can improve the computation of the Stokes multipliers s_2 and s_4 in the preceding example by making the change of variable $x=\xi-\frac{1}{2}\alpha$. This essentially reduces the polynomial $q(x)$ to a single term $(\frac{1}{2}\xi^2)$ and makes the exponential factors $\exp(\alpha\sqrt{n/2})$ and $\exp(-\alpha\sqrt{n/2})$ disappear from the asymptotic representation (4.2). Consequently, both s_2 and s_4 can be computed from the asymptotic behaviour of the coefficients $\tilde{h}_{1n}$ of the modified formal solution $\tilde{f}_1$, regardless of the value of α.

However, in the case of higher-order equations, the appearance of exponentials with different orders of growth in the asymptotic representations of the coefficients $\hat{h}_{jn}$, as in (4.2), cannot usually be avoided.

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